ON A CHARACTERISTIC PROPERTY OF STATIONARY STREAMS WITHOUT AFTEREFFECT

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Reprinted from the
JOURNAL OF APPLIED PROBABILITY
Vol. 2, No. 2, pp, 455–461
December 1965

J. Appl. Prob. **2**, 455–461 (1965)
Printed in Israel

ON A CHARACTERISTIC PROPERTY OF STATIONARY STREAMS WITHOUT AFTEREFFECT

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1. Introduction

Let us consider a stream of homogeneous random events which we call signals and let us denote by $P_k(a,t)$ $(a,t \geqq 0)$ the probability that exactly k signals will appear in the time interval $[a, a+t)$. We assume that the stream is not empty, i.e. the probability $P_0(a,t)$ is not identically equal to 1.

Let us denote by $E(a,t)$ the expected number of signals which appear in the interval $[a, a+t)$. Using Khintchine's [2], [3] terminology, we call a stream finite, if

$$E(t) = E(0,t) < \infty \quad \text{for every} \quad t \geqq 0.$$

For a finite stream $P_\infty(a,t) \equiv 0$, hence

$$\sum_{k=0}^{\infty} P_k(a,t) \equiv 1. \tag{1}$$

We say that the stream is without aftereffect, if its behaviours in disjoint time intervals are mutually independent. A stream without aftereffect is completely defined by the system of the functions $P_k(a,t)$. The function

$$F(t,x) = \sum_{k=0}^{\infty} P_k(a,t)x^k \quad (|x| \leqq 1) \tag{2}$$

is called the generating function of the stream.

A stream without aftereffect is called stationary, if $P_k(a,t)$ is independent of a. For a stationary stream we denote this probability by $P_k(t)$.

Redheffer [4] and Khintchine [2] have found the general form of the finite stationary streams without aftereffect. Khintchine's result can be formulated as follows:

The class of all finite stationary streams without aftereffect coincides with the class of all the streams without aftereffect, the generating function of which can be represented by the following formula:

Received in revised form 9 February 1965.

(3) $$F(t,x) = \exp\{\lambda t[f(x)-1]\},$$

where $\lambda > 0$ and $p_k \geqq 0$, $\sum_{k=1}^{\infty} p_k = 1$ are constants and $f(x) = \sum_{k=1}^{\infty} p_k x^k (|x| \leqq 1)$.

The aim of our note is to give another characterization of this class of streams. It is often pointed out [1], that if a stationary stream without aftereffect satisfies the condition

(4) $$1 - P_0(a,t) - P_1(a,t) = o(t), \quad (t \to 0),$$

then the signals are uniformly distributed in every time interval. In other words, let B be the event that the interval $[a,b)$ contains n signals and let S be one of them, taken at random; let A be the event that S occurs in the interval $[a',b')$, where $a \leqq a' < b' \leqq b$. Then the conditional probability $P(A|B)$ is given by

$$P(A|B) = (b'-a')/(b-a).$$

It turns out that this property, which for the sake of brevity we call property (U), is possessed by every stationary stream without aftereffect, i.e. without the additional condition (4). Moreover, the following proposition holds:

Theorem. A finite stream without aftereffect is stationary if and only if it possesses the property (U).

2. Auxiliary propositions

We will need the expression of the property (U) in terms of the probabilities $P_k(a,t)$. For a stream without aftereffect we obtain the following formula for the probability $P(AB)$ of the joint occurence of A and B:

$$P(AB) = \sum_{r=1}^{n} \frac{r}{n} P_r(a', b'-a') \sum_{s=0}^{n-r} P_s(a, a'-a) P_{n-r-s}(b', b-b').$$

Since $P(B) = P_n(a, b-a)$, the property (U) can be expressed by the relation

$$P_n(a,b-a)(b'-a')/(b-a) = \sum_{r=1}^{n} \frac{r}{n} P_r(a', b'-a') \sum_{s=0}^{n-r} P_s(a, a'-a) P_{n-r-s}(b', b-b')$$

where $0 \leqq a \leqq a' < b' \leqq b$.

Substituting $a = a' = 0$ and then replacing b' and b by a and $a+t$ respectively in (5) we obtain

(6) $$P_n(0, a+t)a/(a+t) = \sum_{r=1}^{n} \frac{r}{n} P_r(0,a) P_{n-r}(a,t).$$

Substituting $a = 0$ and then replacing a' by a and b, b' by $a+t$ we obtain

(7) $$P_n(0, a+t)t/(a+t) = \sum_{r=1}^{n} \frac{r}{n} P_r(a,t) P_{n-r}(0,a).$$

If the stream is also stationary, then

$$\sum_{s=0}^{n-r} P_s(0,a)P_{n-r-s}(b,t-b) = P_{n-r}(t-b+a);$$

in this case formula (5) takes the form

$$P_n(t)\tau/t = \sum_{r=1}^{n} \frac{r}{n} P_r(\tau)P_{n-r}(t-\tau), \tag{8}$$

where $t = b - a$, $\tau = b' - a'$.

Lemma 1. For every finite stream without aftereffect, possessing the property (U), there exists a positive constant λ such, that for every $t \geqq 0$ we have

$$E(t) = \lambda t. \tag{9}$$

Proof. By our assumption, the relation (6) holds, hence also

$$\frac{a}{a+t} \sum_{n=1}^{\infty} nP_n(0,a+t) = \sum_{n=1}^{\infty} \sum_{r=1}^{n} rP_r(0,a)P_{n-r}(a,t).$$

In view of (1), when considering the right part of this equality as a double series with non-negative terms, we obtain

$$\frac{a}{a+t} \sum_{n=1}^{\infty} nP_n(0,a+t) = \sum_{n=1}^{\infty} nP_n(0,a),$$

or

$$E(a+t)/(a+t) = E(a)/a,$$

where $E(t)$ is a bounded function, because the stream is finite. The latter equation shows that the ratio $E(t)/t$ is independent of t, which proves the validity of the formula (9). Since the stream is not empty, $\lambda > 0$.

Lemma 2. If a stream without aftereffect possesses the property (U) and

$$P_0(0,t) > 0 \tag{10}$$

for $t \geqq 0$, then the following formula holds for every natural n:

$$P_n(a,t) = tF_n(t)P_0(a,t), \tag{11}$$

where $F_n(t)$ is a polynomial (including $F_n(t) \equiv 0$).

Proof. Let us put $P(t) = P_0(0,t)$. Since the stream is without aftereffect, we have in view of (10)

$$P_0(a,t) = P(a+t)/P(a). \tag{12}$$

The formula (11) will be proved by induction. The relation (6) for $n = 1$ becomes

$$P_1(0, a+t)a/(a+t) = P_1(0,a)P_0(a,t),$$

hence, by (12),

$$P_1(0, a+t)/[(a+t)P(a+t)] = P_1(0,a)/[aP(a)].$$

Thus we conclude that

$$P_1(0,t) = tp_1P(t), \tag{13}$$

where $p_1 \geqq 0$ is a constant. On the other hand, we get from (7) for $n = 1$ that

$$P_1(0, a+t)t/(a+t) = P_1(a,t)P_0(0,a),$$

therefore by virtue of (12) and (13), we obtain

$$P_1(a,t) = tp_1P_0(a,t),$$

which proves the validity of (11) for $n = 1$.

Let us put $n+1$ instead of n in relation (6). Then

$$P_{n+1}(0, a+t)a/(a+t) = \frac{1}{n+1}\sum_{r=1}^{n} rP_r(0,a)P_{n+1-r}(a,t) + P_{n+1}(0,a)P_0(a,t). \tag{14}$$

When assuming that for $1 \leqq r \leqq n$ the formula (11) holds, we have

$$\begin{aligned} P_r(0,a)P_{n+1-r}(a,t) &= atF_r(a)F_{n+1-r}(t)P_0(0,a)P_0(a,t) \\ &= atF_r(a)F_{n+1-r}(t)P(a+t), \end{aligned}$$

where F_r and $F_{n+1-r}(1 \leqq r \leqq n)$ are polynomials. Let us put

$$F_{n+1}(t) = P_{n+1}(0,t)/[tP(t)], \tag{15}$$

then we obtain from (14) the following relation:

$$F_{n+1}(a+t) - F_{n+1}(a) = \frac{t}{n+1}\sum_{r=1}^{n} rF_r(a)F_{n+1-r}(t). \tag{16}$$

We conclude from here, that $F_{n+1}(t)$ is differentiable for every $t > 0$ and that the derivative $F'_{n+1}(t)$ is a polynomial. Hence, $F_{n+1}(t)$ is also polynomial. The equality (15) can be rewritten as

$$P_{n+1}(0,t) = tF_{n+1}(t)P_0(0,t). \tag{17}$$

Let us put $n+1$ instead of n in (7). Because of (17) we obtain

$$P_{n+1}(a,t) = tP_0(a,t)\left[F_{n+1}(a+t) - \frac{a}{n+1}\sum_{r=1}^{n} rF_r(t)F_{n+1-r}(a)\right].$$

Since formula (16) remains valid if we interchange the roles of a and t, we conclude from the latter relation that

$$P_{n+1}(a,t) = tF_{n+1}(t)P_0(a,t),$$

which proves the lemma.

3. The proof of the theorem

Necessity. Let the stream be stationary and without aftereffect. Then, by Khintchine's theorem its generating function has the form (3). It follows from this that for every τ, t $(0 \leqq \tau \leqq t)$ and x $(|x| \leqq 1)$ we have:

$$\begin{aligned} F(t,x) &= F(\tau,x)F(t-\tau,x), \\ F'_x(t,x) &= F(t,x)f'(x)\lambda t \end{aligned}$$

and

$$F'_x(\tau,x) = F(\tau,x)f'(x)\lambda\tau,$$

which yield the relation

$$F'_x(t,x)\tau/t = F'_x(\tau,x)F(t-\tau,x).$$

Differentiating $n-1$ times with respect to x, we obtain the formula

$$F_x^{(n)}(t,x)\tau/t = \sum_{k=0}^{n-1} \binom{n-1}{k} F_x^{(k+1)}(\tau,x)F_x^{(n-1-k)}(t-\tau,x).$$

Hence, putting $k+1=r$, $x=0$, we get the equality

$$\frac{1}{n!}F_x^{(n)}(t,0)\tau/t = \sum_{r=1}^{n} \left[F_x^{(r)}(\tau,0)/r!\right] \left[F_x^{(n-r)}(t-\tau,0)/(n-r)!\right] \frac{r}{n},$$

which, by the definition (2) of the generating function $F(t,x)$, coincides with formula (8). Thus the stream possesses the property (U).

Sufficiency. Let a stream without aftereffect possess the property (U). It follows from one of Khintchine's lemmas ([3], Lemma 8) that if the stream is without aftereffect and the function $E(t)$ is continuous, then the probability $P_0(0,t)$ satisfies the condition (10). Hence, by virtue of Lemmas 1 and 2, the probabilities $P_n(a,t)$ are given by (11). Therefore, in view of (1), we get

$$P_0(a,t) = 1 \Big/ \left[1 + t \sum_{n=1}^{\infty} F_n(t)\right].$$

Thus the function $P_0(a,t)$ and, together with it, the functions $P_n(a,t)$ are independent of a. Hence, the stream is stationary.

4. The general form of the generating function

In conclusion we will show that the formula (3) may be obtained in what seems to be a simpler way than in [2].

If $F(t,x)$ is the generating function of a stationary stream without aftereffect, then for every t, τ and x $(t,\tau \geqq 0, |x| \leqq 1)$ we have

$$F(t+\tau,x) = F(t,x)F(\tau,x).$$

It follows that $F(t,x)$ must be of the form

$$F(t,x) = \exp[-tg(x)], \tag{18}$$

where $g(x)$, because of (2), can be represented as

$$g(x) = \lambda - \sum_{k=1}^{\infty} \lambda_k x^k, \quad (|x| \leqq 1). \tag{19}$$

For a non-empty stream we have

$$F(t,0) = P_0(t) < 1,$$

hence

$$g(0) = \lambda > 0.$$

Next, because of (1), we have $F(t,1) = 1$, so that $g(1) = 0$, i.e.,

$$\lambda = \sum_{k=1}^{\infty} \lambda_k. \tag{20}$$

In order to obtain formula (3) it is sufficient to show that $\lambda_k \geqq 0$.

It follows from (18) that

$$F'_x(t,x) = tF(t,x)g'(x).$$

Differentiating $k-1$ times with respect to x, we get the equation

$$\frac{1}{k!} F_x^{(k)}(t,x) = \frac{t}{k} \sum_{i=0}^{k-1} (i+1) \left[g^{(i+1)}(x)/(i+1)! \right] \left[F_x^{(k-1-i)}(t,x)/(k-1-i)! \right] \tag{21}$$

By (2) and (19) we have for every i

$$\lambda_i = g^{(i)}(0)/i!, \quad P_i(t) = F_x^{(i)}(t,0)/i!$$

Therefore, putting $x = 0$ in (21), we get

$$P_k(t)/t = \frac{1}{k} \sum_{i=1}^{k} i\lambda_i P_{k-i}(t).$$

Since $P_0(0+) = 1$ and $P_k(0+) = 0$ for $k \geqq 1$, we obtain

$$\lim_{t \to 0} [P_k(t)/t] = \lambda_k \geqq 0.$$

Denoting $p_k = \lambda_k/\lambda$, we then obtain, by (20), formula (3).

The generating function $F(t,x)$ may also be written as

$$F(t, x) = \prod_{k=1}^{\infty} \phi_k(t, x), \tag{22}$$

where

$$\phi_k(t, x) = \exp[t\lambda_k(x^k - 1)].$$

In Khintchine's terminology [2], a stationary stream without aftereffect $x(t)$ is called simple, if it satisfies condition (4). It is easy to see that $\phi_k(t, x)$ is the generating function of the stream $y(t) = kx(t)$, where $x(t)$ is a simple stream. Then the formula (22) may be interpreted as follows:

The class of all finite stationary streams without aftereffect coincides with the class of all sums of mutually independent multiples of simple streams.

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