A GENERALISATION OF ERLANG'S FORMULAS IN QUEUEING THEORY

BY
D. MEJZLER

Reprinted from the
JOURNAL OF APPLIED PROBABILITY
Vol. 5, No. 1, pp. 143–157
April 1968

J. Appl. Prob. **5**, 143–157 (1968)
Printed in Israel

A GENERALISATION OF ERLANG'S FORMULAS IN QUEUEING THEORY

D. MEJZLER, *The Hebrew University of Jerusalem*

1. Introduction

We follow the terminology of Khintchine's monograph [4].

Let us consider a random stream of calls entering a service system which consists of n lines. The service durations are assumed to be identically distributed random variables which are independent both of each other and of the course of the incoming stream.

Denote by $P_k(t)$ the probability that at the moment t the system is in state k, i.e. exactly k lines are occupied. If the incoming stream is simple (a Poisson stream) then the limits

$$P_k = \lim_{t\to\infty} P_k(t), \qquad (k = 0, 1, \cdots, n)$$

exist and are independent of the initial probabilities $P_k(0)$, $(k = 0, 1, \cdots, n)$. These limits are given by the formulas

$$P_k = \frac{(\lambda a)^k}{k!} \Big/ \sum_{i=0}^{n} \frac{(\lambda a)^i}{i!}, \qquad (k = 0, 1, \cdots, n), \tag{1}$$

where λ is the intensity of the stream and a is the mean service length of one call.

Formulas (1) were first obtained by Erlang [1] under the assumption that the duration of the service is exponentially distributed; however, it was later shown ([2], [5] and [7]) that they remain valid for an arbitrary distribution of the service time.

Khintchine has shown [4] that (1) remains true also for a system with infinitely many lines, i.e. for $n = \infty$,

$$P_k = e^{-\lambda a} \frac{(\lambda a)^k}{k!}, \qquad (k = 0, 1, \cdots). \tag{2}$$

The aim of the present note is to show that if the service length is distributed exponentially, then Erlang's problem can be solved rather easily even in the

Received 10 May 1967.

somewhat more general case, where the incoming stream is an arbitrary (not necessarily orderly) stationary stream without after-effects. In this more general case the formulas for the limit probabilities P_k assume a more complicated form and they coincide with the formulas (1) only for a single-line system. Moreover, for a single-line system formulas (1) turn out to be valid for an arbitrary stationary stream without after-effects and for an arbitrary distribution of the service length.

In the last sections we consider anew a system with infinitely many lines. This was the subject of our note [6]. Unfortunately, the final formulas of [6] are vitiated by a gross error which slipped into our reasoning*. Correcting the final formulas we also give necessary and sufficient conditions in order that all the limit probabilities P_k be positive and satisfy the condition $\sum_{k=0}^{\infty} P_k = 1$.

2. Exponentially distributed service length; finite number of lines

Let a calling moment be any moment at which calls occur. It is known [4] that the calling moments of a stationary stream without after-effects constitute a simple stream; we shall denote its intensity by λ. The number of calls occurring at a calling moment is a random variable, assuming the value i ($i = 1, 2, \cdots$) with probability p_i ($p_i \geqq 0$, $\sum_{i=1}^{\infty} p_i = 1$), independently of what the course of the stream is up to that moment.

Denote by $v_k(t)$ ($v_k^*(t)$) the probability that a time interval of length t contains k calls (calling moments). Then

$$v_0(t) = v_0^*(t) = e^{-\lambda t},$$

$$v_1(t) = v_1^*(t) \cdot p_1 = \lambda t e^{-\lambda t} p_1$$

and for $k \geqq 2$, since the stream of calling moments is orderly, we obtain

$$v_k(t) = v_1^*(t) p_k + o(t), \qquad (t \to 0).$$

Thus for $k \geqq 1$ and $\Delta t > 0$

$$v_k(t) = \lambda \Delta t p_k + o(\Delta t), \qquad (\Delta t \to 0). \tag{3}$$

Suppose that a stream, as described above. enters a service system with n lines, and let the service length η have the distribution

$$F(x) = P(\eta \leqq x) = 1 - e^{-x/a}, \tag{4}$$

where a is obviously the expected value of η. Denote by $N(t)$ the number of lines occupied at the moment t. Under our assumptions $N(t)$ is a homogeneous Markov process, and for every $t \geqq 0$, $\Delta t > 0$ and $k (0 \leqq k \leqq n)$,

---

*) I am indebted to Prof. P. Naor of the Israel Institute of Technology, Haifa, for calling my attention to a counter-example to the formulas of [6].

$$P_k(t+\Delta t) = \sum_{i=0}^{n} P_i(t) P_{ik}(\Delta t), \tag{5}$$

where $P_{ik}(\Delta t)$ is the probability that during the time Δt the system will pass from state i to state k.

The asymptotic behaviour of the transition probabilities $P_{ik}(\Delta t)$ for $\Delta t \to 0$ may be determined in exactly the same way as in the case of a simple incoming stream. But whereas in the latter case one has, by virtue of the orderliness, for every i and k, $|i-k|>1$

$$P_{ik}(\Delta t) = o(\Delta t), \qquad (\Delta t \to 0),$$

we now have, because of (3), for $\Delta t \to 0$ the following approximations (to within $o(\Delta t)$):

$$\begin{aligned}
P_{kk}(\Delta t) &\approx 1-(\lambda + k/a)\Delta t, \qquad (0 \leqq k \leqq n-1),\\
P_{nn}(\Delta t) &\approx 1 - \frac{n}{a}\Delta t,\\
P_{k,k-1}(\Delta t) &\approx \frac{k}{a}\Delta t,\\
P_{k+i,k}(\Delta t) &\approx 0, \qquad (i \geqq 2)\\
P_{k,k+i}(\Delta t) &\approx \lambda p_i \Delta t, \qquad (1 \leqq i \leqq n-1-k),\\
P_{kn}(\Delta t) &\approx \lambda w_{n-k}\Delta t, \qquad (0 \leqq k \leqq n-1),
\end{aligned}$$

where w_i is the probability that at a calling moment there occur at least i calls, i.e.

$$w_i = \sum_{s=i}^{\infty} p_s. \tag{6}$$

These relations, together with (5), lead to the following system of $(n+1)$ differential equations for the $(n+1)$ unknown functions $P_k(t)$ $(k=0,1,\cdots,n)$:

$$\begin{aligned}
P_0'(t) &= -\lambda P_0(t) + P_1(t)/a,\\
P_k'(t) &= \lambda \sum_{i=0}^{k-1} P_i(t) p_{k-i} - (\lambda + k/a)P_k(t) + \frac{k+1}{a} P_{k+1}(t), \qquad (0<k<n), \qquad (7)\\
P_n'(t) &= \lambda \sum_{i=0}^{n-2} P_i(t) w_{n-i} + \lambda P_{n-1}(t) - \frac{n}{a} P_n(t).
\end{aligned}$$

The process $N(t)$ is obviously transitive. Therefore, by a well-known theorem of Markov the existence and ergodicity of the limit probabilities P_k are assured. To compute them, let us pass to the limit (for $t \to \infty$) in (7). This leads to the following system of linear equations:

$$0 = -\lambda P_0 + P_1/a,$$

$$0 = \lambda \sum_{i=0}^{k-2} P_i p_{k-i} - (\lambda + k/a)P_k + \frac{k+1}{a} P_{k+1}, \qquad (0 < k < n), \tag{8}$$

$$0 = \lambda \sum_{i=0}^{n-2} P_i w_{n-i} + \lambda P_{n-1} - \frac{n}{a} P_n.$$

The last equation, which violates the symmetry of the system, is a linear combination of the preceding equations and can therefore be omitted. The remaining n equations, together with the "normalising" condition

$$\sum_{k=0}^{n} P_k = 1, \tag{9}$$

uniquely determine the required numbers P_k. Adding up the first m equations of (8) for $m = 1, \cdots, n$, we obtain

$$-\lambda a \sum_{i=1}^{k-1} P_i w_{k-i} + kP_k = \lambda a w_k P_0, \qquad (1 \leqq k \leqq n). \tag{10}$$

This system of equations is such that the first k equations suffice to determine the ratio P_k/P_0. Putting, for $k \geqq 1$,

$$\delta_k = \lambda a \begin{vmatrix} 1 & 0 & 0 \cdots\cdots 0 & w_1 \\ -\lambda a w_1 & 2 & 0 \cdots\cdots 0 & w_2 \\ -\lambda a w_2 & -\lambda a w_1 & 3 \cdots\cdots 0 & w_3 \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ -\lambda a w_{k-2} & -\lambda a w_{k-3} & \cdots\cdots\ k-1 & w_{k-1} \\ -\lambda a w_{k-1} & -\lambda a w_{k-2} & \cdots\cdots\ -\lambda a w_1 & w_k \end{vmatrix}$$

we get

$$P_k = \frac{\delta_k}{k!} P_0.$$

Hence, taking $\delta_0 = 1$ and using (9) we obtain

$$P_k = \frac{\delta_k}{k!} \Big/ \sum_{i=0}^{n} \frac{\delta_i}{i!}, \qquad (k = 0, 1, \cdots, n). \tag{11}$$

These formulas solve our problem. However, in practice it seems more convenient to compute the ratio P_k/P_0 from the recursive formula

$$kP_k = \lambda a \sum_{i=0}^{k-1} P_i w_{k-i}, \qquad (k = 1, 2, \cdots, n), \tag{12}$$

which is merely another form of (10). Since P_k/P_0 is independent of n, the transition from n to $(n+1)$ requires only the computation of P_{n+1}/P_0 and the redetermination of P_0 from (9).

The numerator of (11) may be expressed also in a different way. It is easy to construct a generating function for the determinants δ_k. Let

$$f(x) = \sum_{k=1}^{n} \frac{w_k}{k} x^k, \tag{13}$$

$$G(x) = e^{\lambda a f(x)}. \tag{14}$$

We shall show that

$$\delta_k = G^{(k)}(0),$$

or, in other words,

$$P_k = \frac{G^{(k)}(0)}{k!} P_0, \qquad (k = 0, 1, \cdots, n). \tag{15}$$

We have

$$G'(x) = \lambda a G(x) f'(x).$$

Differentiating this identity $(m-1)$ times, we get

$$G^{(m)}(x) = \lambda a \sum_{i=0}^{m-1} \binom{m-1}{i} G^{(i)}(x) f^{(m-i)}(x).$$

Hence

$$m \frac{G^{(m)}(0)}{m!} = \lambda a \sum_{i=0}^{m-1} \frac{G^{(i)}(0)}{i!} \cdot \frac{f^{(m-i)}(0)}{(m-1-i)!}.$$

But from the definition (13) of $f(x)$ it follows that

$$f^{(m-i)}(0) = (m-1-i)! w_{m-i}.$$

Therefore

$$m \frac{G^{(m)}(0)}{m!} = \lambda a \sum_{i=0}^{m-1} \frac{G^{(i)}(0)}{i!} w_{m-i}.$$

Assuming the validity of (15) for $k = 1, \cdots, m-1$ we conclude, because of (12), that (15) holds also for $k = m$. Thus formula (11) can also be written in the form

$$P_k = \frac{G^{(k)}(0)}{k!} \Big/ \sum_{i=0}^{n} \frac{G^{(i)}(0)}{i!}, \qquad (k = 0, 1, \cdots, n), \tag{16}$$

where $G(x)$ is defined by (13) and (14),

In particular, when, in addition to the other assumptions we also assume that the incoming stream is orderly, then $w_k = 0$ for $k \geqq 2$, the function $G(x)$ assumes the form

$$G(x) = e^{\lambda a x}$$

and formula (16) coincides with (1).

3. A single-line system

For $n = 1$ formulas (11) assume the form

$$P_0 = 1/(1 + \lambda a), \qquad P_1 = \lambda a/(1 + \lambda a), \tag{17}$$

which is the same as (1). We shall now show that (1) also holds for an arbitrary distribution of the service length η.

Let n and $N(t)$ be as above. Since η is the same for all calls of the stream and does not depend on its course, it is evident that the probability behaviour of $N(t)$ does not change in any way if the stream passes through a filter which does not admit more than m calls at a time, where $m \geqq n$. More precisely, if for a stationary stream without after-effects the number of calls occuring at a calling moment has the distribution $\{p_i\}$, then in investigating the behaviour of the random function $N(t)$ we may assume without loss of generality that $p_{n+i} = 0$ for $i \geqq 1$ and therefore $p_n = w_n$. (We have already made use of this in determining the asymptotic behaviour of the probabilities $P_{kn}(\Delta t)$.) It follows that if the system has a single line, then for a given η, however distributed, the state $N(t)$ of the system is completely determined by the stream of calling moments, and does not depend on the number of calls occuring at one calling moment or another. Since the stream of calling moments is Poisson, and since for a Poisson stream formula (1) holds for any distribution of η, it follows that (17) is valid for any η and any stationary stream without after-effects.

Formula (17) can be extended to a slightly more general scheme. Let a finite or denumerable set of mutually independent stationary streams without after-effects $x_i(t)$ with parameters λ_i $(i = 1, 2, \cdots)$ enter a single-line system. Let us assume that the distribution of the service length of a call is the same for calls belonging to the same stream but may depend on the index i. Let us denote by a_i the mean service length for a call belonging to the ith stream. Let us also assume that $a_i < \infty$ $(i = 1, 2, \cdots)$, $\sum_{i=1}^{\infty} \lambda_i < \infty$, $\sum_{i=1}^{\infty} \lambda_i a_i < \infty$. Then the total stream

$$x(t) = \sum_{i=1}^{\infty} x_i(t)$$

is evidently a stationary stream without after-effects whose parameter is

$$\lambda = \sum_{i=1}^{\infty} \lambda_i,$$

and the mean service length of a call in the total stream is

$$a = \frac{1}{\lambda} \sum_{i=1}^{\infty} \lambda_i a_i.$$

Clearly, also in this case the limit probabilities P_0 and P_1 exist and are given by (17).

Remark. Let M and N be two systems consisting of m and n lines, respectively, where $m < n$. Suppose a stream enters both systems simultaneously, let η be the same for both systems, and let $P_k^{(m)}$ — $(P_k^{(n)})$ — be the probability that system M — (N) — be in state k. Formulas (11) show that if η is distributed exponentially, then for every k $(0 \leqq k \leqq m)$

$$P_k^{(m)} = P_k^{(n)} \Big/ \sum_{i=0}^{m} P_i^{(n)}. \tag{18}$$

The expression on the right hand side is the conditional probability that system N is in state k provided it is in one of the states $0, 1, \cdots, m$. Therefore, from physical considerations, it seems plausible that (18) should be true independently of the distribution of η. If this is so, then for any η and any n one should have

$$P_1^{(n)} = \lambda a P_0^{(n)}.$$

4. A system with infinitely many lines; $P_0(0) = 1$

Let a stationary stream without after-effects, determined by the quantities λ and p_i, enter a system consisting of $n = \infty$ lines. For the sake of simplicity let us assume that the stream begins at the moment $t_0 = 0$, so that at that moment all lines are vacant: $P_0(0) = 1$. (In the next section we shall show that the limit probabilities P_k are independent of the initial distribution $P_k(0)$.) Let us put

$$F(x) = P(\eta \leqq x), \qquad H(x) = 1 - F(x). \tag{19}$$

We shall first find the generating function of the probabilities $P_k(t)$:

$$G(t, x) = \sum_{k=0}^{\infty} P_k(t) x^k, \qquad (|x| \leqq 1). \tag{20}$$

Consider a calling moment ξ in the interval $[0, t)$, at which moment there occur $(m + i)$ calls. Let these calls, arranged in some arbitrary way, be denoted

by C_r $(r = 1,2,\cdots,m+i)$ and let us compute the probability W that the service of the first m calls ends before the moment t and the service of the remaining i calls does not end before t. Denoting by η_r the service duration of the call C_r, we can write the required probability in the form

$$W = P(\xi + \eta_r \leqq t,\ r = 1,\cdots,m;\ \xi + \eta_r > t,\ r = m+1,\cdots,m+i).$$

Under our assumptions the quantities η_r are mutually independent and identically distributed according to (19). The stream of calling moments is Poisson, so that ξ is uniformly distributed in the interval $[0,t)$, whatever the number of calling moments occurring in this interval (see [3]). Moreover, ξ is independent of the η_r. Consequently

$$W = \frac{1}{t} \int dx dF(x_1) \cdots dF(x_{m+i}),$$

where the domain of integration is defined by the inequalities*

$$0 \leqq x \leqq t;\ 0 \leqq x_r \leqq t - x,\ r = 1,\cdots,m;\ x_r > t - x,\ r = m+1,\cdots,m+i.$$

One obtains

$$W = \frac{1}{t} \int_0^t H^i(x) F^m(x) dx. \tag{21}$$

Now let α be the number of those among the calls occurring at a calling moment in the interval $[0,t)$, whose service does not end before the moment t. In view of (21) it is easy to see that for any $t > 0$ and $i \geqq 0$

$$W(t,i) = P(\alpha = i) = \frac{1}{t} \sum_{m=0}^{\infty} \binom{m+i}{i} p_{m+i} \int_0^t H^i(x) F^m(x) dx, \tag{22}$$

where $p_0 = 0$. Since the probability that a finite time interval contains only finitely many calls is 1, it follows that for any finite t

$$P(\alpha = \infty) = 0,$$

so that

$$\sum_{i=0}^{\infty} W(t,i) \equiv 1, \tag{23}$$

which could also be verified directly from (22). The generating function of α has the form

---

* The error of [6] consisted in neglecting the mutual dependence of the sums $(\xi + \eta_r)$.

$$g(t,x) = \sum_{i=0}^{\infty} W(t,i)x^i, \qquad (|x| \leqq 1). \tag{24}$$

Let us now consider any r calling moments in the interval $[0,t)$ and let β_r be the total number of those among the calls occuring at these moments, whose service does not terminate before t. Put

$$W_r(t,k) = P(\beta_r = k), \qquad (r \geqq 1),$$

$$W_0(t,0) = 1, \quad W_0(t,k) = 0, \qquad (k \geqq 1),$$

and

$$g_r(t,x) = \sum_{k=0}^{\infty} W_r(t,k)x^k, \qquad (|x| \leqq 1). \tag{25}$$

Since β_r is the sum of r mutually independent random variables α, we have for any $r \geqq 0$

$$g_r(t,x) \equiv g^r(t,x). \tag{26}$$

On the other hand, it is clear that for any $t > 0$ and $k \geqq 0$

$$P_k(t) = \sum_{r=0}^{\infty} e^{-\lambda t}\frac{(\lambda t)^r}{r!} W_r(t,k). \tag{27}$$

Therefore, in virtue of (20),

$$G(t,x) = \sum_{k=0}^{\infty} x^k \sum_{r=0}^{\infty} e^{-\lambda t}\frac{(\lambda t)^r}{r!} W_r(t,k)$$

$$= e^{-\lambda t} \sum_{r=0}^{\infty} \frac{(\lambda t)^r}{r!} \sum_{k=0}^{\infty} W_r(t,k)x^k.$$

Hence, by (25) and (26) it follows that

$$G(t,x) = \exp\{\lambda t[g(t,x) - 1]\}, \qquad (|x| \leqq 1), \tag{28}$$

where $g(t,x)$ is given by (22) and (24). Since, by (23), $g(t,1) = 1$, we have, using (20) and (28),

$$\sum_{k=0}^{\infty} P_k(t) \equiv 1. \tag{29}$$

To investigate the behaviour of the $P_k(t)$ as $t \to \infty$, we require explicit expressions for them. From (28) we get

$$P_0(t) = G(t,0) = \exp\{\lambda t[W(t,0) - 1]\}, \tag{30}$$

which can also be written as

$$P_0(t) = \exp\left\{ -\lambda \sum_{m=1}^{\infty} p_m \int_0^t [1 - F^m(x)]dx \right\}. \tag{31}$$

From (24), (25) and (26) we conclude that for $r \geqq 1$

$$W_r(t,k) = \Sigma\, W(t,i_1)\cdots W(t,i_r),$$

where the summation is over all ordered r-tuples of non-negative integers $(i_1,\cdots,i_r)$ for which

$$i_1 + \cdots + i_r = k.$$

It follows that for any natural r and k

$$W_r(t,k) = \sum_{s=1}^{k} \binom{r}{s} W^{r-s}(t,0)\ \Sigma^* W(t,i_1)\cdots W(t,i_s), \tag{32}$$

where now Σ^* denotes summation over all ordered s-tuples of positive integers $(i_1,\cdots,i_s)$ for which

$$i_1 + \cdots + i_s = k.$$

Therefore, by (27), when $k \geqq 1$,

$$P_k(t) = \sum_{r=0}^{\infty} e^{-\lambda t}\frac{(\lambda t)^r}{r!} \sum_{s=1}^{k} \binom{r}{s} W^{r-s}(t,0)\, \Sigma^* W(t,i_1)\cdots W(t,i_s)$$

$$= \sum_{s=1}^{k} \frac{(\lambda t)^s}{s!}\, \Sigma^* W(t,i_1)\cdots W(t,i_s) \sum_{r=s}^{\infty} e^{-\lambda t}\, \frac{(\lambda t)^{r-s}}{(r-s)!}\, W^{r-s}(t,0),$$

whence by (30)

$$P_k(t) = P_0(t) \sum_{s=1}^{k} \frac{(\lambda t)^s}{s!}\, \Sigma^* W(t,i_1)\cdots W(t,i_s).$$

Putting

$$A(t,i) = tW(t,i) = \sum_{m=0}^{\infty} \binom{m+i}{i} p_{m+i} \int_0^t H^i(x)F^m(x)dx, \tag{33}$$

we write $P_k(t)$ in the final form

$$P_k(t) = P_0(t) \sum_{s=1}^{k} \frac{\lambda^s}{s!}\, \Sigma^* A(t,i_1)\cdots A(t,i_s). \tag{34}$$

Let $t \to \infty$. We shall now show that under our assumptions all the limits P_k exist and are either all 0, or are all positive and, in the latter case, we have:

$$\int_0^\infty [1 - F^m(x)]dx < \infty, \qquad (m = 1, 2, \cdots), \tag{35}$$

$$A_0 = \sum_{m=1}^{\infty} p_m \int_0^\infty [1 - F^m(x)]dx < \infty, \tag{36}$$

$$A_i = \sum_{m=0}^{\infty} \binom{m+i}{i} p_{m+i} \int_0^\infty H^i(x)F^m(x)dx < \infty, \qquad (i = 1, 2, \cdots), \tag{37}$$

$$P_0 = e^{-\lambda A 0}, \tag{38}$$

$$P_k = P_0 \sum_{s=1}^{k} \frac{\lambda^s}{s!} \Sigma^* A_{i_1} \cdots A_{i_s} \qquad (k = 1, 2, \cdots), \tag{39}$$

$$\sum_{k=0}^{\infty} P_k = 1. \tag{40}$$

From (31) it is evident that $P_0(t)$ is a non-increasing function and therefore the limit P_0 exists. In view of (23), for every $t > 0$ and $i \geqq 1$

$$1 - W(t, 0) \geqq W(t, i), \tag{41}$$

whence by (30) and (33),

$$-\ln P_0(t) \geqq \lambda A(t, i).$$

Hence we conclude, using (34), that for every $k \geqq 1$ the inequalities

$$0 \leqq P_k(t) \leqq P_0(t)\, B[-\ln P_0(t)]$$

hold, where $B(x)$ is a polynomial of degree k. This shows that if $P_0 = 0$ then for every k the limit P_k exists and equals 0.

Let us now assume that $P_0 > 0$. Then by (31) the integrals in (35) exist and the series in (36) converges. Since the functions $\int_0^t [1 - F^m(x)]dx$ are non-decreasing, P_0 is given by (38). From (34) it is seen that the quotient $P_k(t)/P_0(t)$ is non-decreasing. Moreover it is bounded:

$$P_k(t)/P_0(t) < 1/P_0.$$

Consequently, the limit P_k exists and is positive. This implies the convergence of the series in (37) and thus justifies passage to the limit term-by-term in (33). This proves (39).

Let us write (29) in the form

$$\sum_{k=0}^{\infty} [P_k(t)/P_0(t)] = 1/P_0(t).$$

Since the terms of this series are non-decreasing functions, we can pass to the limit term-by-term here too, whence (40).

From (41) it follows that $A_i \leqq A_0$, $(i = 1,2,\cdots)$. Therefore conditions (35) and (36) are necessary and sufficient for the positiveness of the limits P_k. Since for every $t > 0$ and natural m

$$\int_0^t [1 - F^m(x)]dx = \int_0^t H(x)[1 + F(x) + \cdots + F^{m-1}(x)]\,dx \leqq m\int_0^t H(x)dx,$$

conditions (35) and (36) are always satisfied if the mean service length is finite, i.e.,

$$a = \int_0^\infty x dF(x)dx = \int_0^\infty H(x)dx < \infty \tag{42}$$

and the incoming stream is finite, i.e.,

$$\sum_{i=1}^{\infty} ip_i < \infty. \tag{43}$$

Condition (42), being equivalent to (35), is also necessary for $P_k > 0$, while condition (43) is not — for the series (36) may converge even when the stream is not finite.

Let

$$G(x) = \sum_{k=0}^{\infty} P_k x^k, \qquad (|x| \leqq 1), \tag{44}$$

be the generating function of the probabilities P_k. Since for $|x| < 1$ the series (20) converges uniformly in t, $0 \leqq t < \infty$, and the limits P_k exist, we have for $|x| < 1$

$$G(x) = \lim_{t\to\infty} G(t,x).$$

By (24), (28) and (33)

$$G(t,x) = \exp\left\{\lambda \sum_{k=1}^{\infty} A(t,k)x^k - \lambda t[1 - W(t,0)]\right\}.$$

Thus if (35) and (36) are satisfied we obtain, using (30) and (31),

$$G(x) = \exp\left[\lambda(-A_0 + \sum_{k=1}^{\infty} A_k x^k)\right], \qquad (|x| \leqq 1); \tag{45}$$

otherwise $G(x) \equiv 0$.

In particular, if the incoming stream is also orderly, then by (42) $G(x)$ assumes the form

$$G(x) = e^{\lambda a(x-1)}$$

and is the generating function of the probabilities (2).

5. Lack of effect of the initial distribution

In the previous section we have supposed that $P_0(0) = 1$. If the probability distribution of the initial state of the system is arbitrary, then the probability, which was denoted by $P_k(t)$, must now be considered as the transition probability $P_{0k}(t)$, whose limit, P_k, for $t \to \infty$ is given by (38) and (39).

It is easy to show that for every i and k

$$\lim_{t\to\infty} P_{ik}(t) = \lim_{t\to\infty} P_{0k}(t) = P_k . \tag{46}$$

Indeed, let us suppose that $P_i(0) = 1$. Under this assumption let E_{ik} be the event that the system pass from state i to state k during the time t and let B_i be the event that all the i "conversations" which are being conducted at the moment $t_0 = 0$ will be terminated before the moment t. Since

$$E_{ik} = E_{ik}B_i + E_{ik}\bar{B}_i ,$$

we have

$$P_{ik}(t) = P(E_{ik}) = P(B_i)\,P(E_{ik} \mid B_i) + P(E_{ik}\bar{B}_i). \tag{47}$$

Clearly

$$P(E_{ik} \mid B_i) = P_{0k}(t) = P_k + o(1), \qquad (t \to \infty). \tag{48}$$

On the other hand we have

$$P(E_{ik}\bar{B}_i) \leqq P(\bar{B}_i) \leqq iP(\eta > t);$$

hence, using (42), we get

$$P(E_{ik}\bar{B}_i) = o(1), \quad P(B_i) = 1 + o(1), \qquad (t \to \infty). \tag{49}$$

Thus (46) follows from (47), (48) and (49).

Let $\{P_i(0)\}$ be an arbitrary distribution of the initial state of the system. Since $\sum_{i=0}^{\infty} P_i(0) = 1$, then the series

$$P_k(t) = \sum_{i=0}^{\infty} P_i(0)P_{ik}(t)$$

converges uniformly in $0 \leqq t < \infty$ and by (46) we get

$$P_k = \lim_{t\to\infty} P_k(t)$$

whatever the initial probabilities $P_i(0)$.

6. Examples

1. Let the service duration η be bounded and let $\bar{a}$ be its least upper bound, i.e.

$$P(\eta \leqq \bar{a}) = 1, \quad P(\eta > \bar{a} - \varepsilon) > 0 \text{ for } \varepsilon > 0. \tag{50}$$

Clearly, whatever the distribution $F(x)$ of η, we get for every m

$$\int_0^{\infty} [1 - F^m(x)]dx = \int_0^{\bar{a}} [1 - F^m(x)]dx \leqq \bar{a}.$$

Thus condition (36) is always satisfied, whatever the distribution $\{p_i\}$.

The random state function $N(t)$ is stationary for $t \geqq \bar{a}$, since by (31), (34) and (50) we have $P_k(t) = P_k$ for $t \geqq \bar{a}$.

2. Let the service duration be a constant quantity a, i.e.

$$P(\eta = a) = 1,$$

and let $t \geqq a$. Then for every natural i and m

$$\int_0^t H^i(x)dx = a, \qquad \int_0^t H^i(x)F^m(x)dx = 0,$$

whence

$$A_0 = \int_0^t [1 - F^m(x)]dx = a, \qquad A(t,i) = ap_i, \qquad (i = 1,2,\cdots). \tag{51}$$

Thus we have for $t \geqq a$

$$P_0(t) = P_0 = e^{-\lambda a}$$

$$P_k(t) = P_k = e^{-\lambda a} \sum_{s=1}^{k} \frac{(\lambda a)^s}{s!} \Sigma^* p_{i_1} \cdots p_{i_s}, \qquad (k = 1,2,\cdots),$$

where the summation in Σ^* is as in (32). These probabilities are always positive whatever the distribution $\{p_i\}$.

One can arrive at the last formulas also in a different way. In the present case, the system is in state k at the moment t ($t \geqq a$) if and only if the interval $[t - a, t)$ contains exactly k calls. Therefore, denoting by $v_k(t)$ the probability that an interval of length t contain k calls, we deduce that for $t \geqq a$

$$P_k(t) = P_k = v_k(a).$$

It is known [4] that for a stationary stream without after-effects the generating function of the probabilities $v_k(t)$ has the form

$$\Phi(t,x) = \sum_{k=0}^{\infty} v_k(t)x^k = \exp\{\lambda t[f(x) - 1]\}, \qquad (|x| \leqq 1),$$

where $f(x) = \sum_{i=1}^{\infty} p_i x^i$. In view of (45) and (51) it is easy to see that for $t \geqq \bar{a}$ one has the identity

$$G(t, x) = G(a, x) = G(x) = \Phi(a, x),$$

where $G(t, x)$ and $G(x)$ are defined by (20) and (44).

3. Let the service duration be distributed exponentially according to (4). Then for any $i \geqq 1$ and $m \geqq 0$

$$\int_0^{\infty} H^i(x) F^m(x) dx = a \Big/ \left[i \binom{m+i}{i} \right].$$

Because of (36) and (38)

$$P_0 = \exp\left(-\lambda \sum_{i=1}^{\infty} p_i \sum_{m=1}^{i} \frac{1}{m} \right); \tag{52}$$

by (37) and (39)

$$P_k = P_0 \sum_{s=1}^{k} \frac{(\lambda a)^s}{s!} \Sigma^* \frac{w_{i_1}}{i_1} \cdots \frac{w_{i_s}}{i_s}, \qquad (k = 1, 2, \cdots),$$

where w_i are defined by (6).

Suppose, e.g.,

$$p_i = \frac{1}{i(i+1)}, \qquad (i = 1, 2, \cdots),$$

then the stream is not finite. Nevertheless, the series (52) converges and all the P_k are positive.

Acknowledgement

I would like to thank Professor A. Dvoretzky of the Hebrew University of Jerusalem for reading the manuscript and for his helpful comments.

References

[1] ERLANG, A. K. (1917–18) Solution of some problems in the theory of probabilities of significance in automatic telephone exchanges. *Post Office Electrical Engineer's Journal* **10,** 189–197.

[2] FORTET, R. (1950) *Calcul de Probabilités.* Centre National de la Recherche Scientifique, Paris.

[3] GNEDENKO, B. V. (1962) *The Theory of Probability.* Chelsea Publishing Company, New York.

[4] KHINTCHINE, A. YA. (1960) *Mathematical Methods in the Theory of Queueing.* Charles Griffin , London.

[5] KHINTCHINE, A. YA. (1962) Erlang's formulas in the theory of mass service. *Theor. Probability Appl.* **7,** No. 3, 320–325.

[6] MEJZLER. D. (1965) A note on Erlang's formulas. *Israel J. Math.* **3,** No. 3. 157–162.

[7] SEVAST'YANOV B. A. (1957) An ergodic theorem for Markov processes and its applications to telephone systems with refusals. *Theor. Probability Appl.* **2,** No. 1. 104–112.